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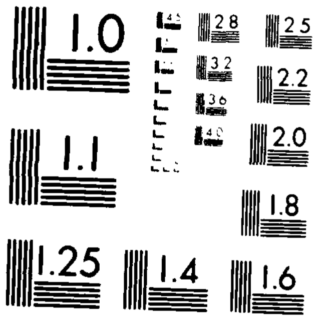
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TECHNICAL REPORT ARBRL-TR-02450

A METHOD OF EVALUATING LAPLACE TRANSFORMS WITH SERIES OF COMPLETE OR INCOMPLETE BETA FUNCTIONS

Alexander S. Elder
Emma M. Wineholt

December 1982

US ARMY ARMAMENT RESEARCH AND DEVELOPMENT COMMAND
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In a previous paper factorial series were used to calculate ordinary and modified Bessel functions of the second kind. In the present paper the factorial series is generalized so that Laplace integrals in which the integrand has a branch point at the origin are represented by a series of beta functions. To effect the required transformation, formulas for calculating Stirling numbers of fractional order were derived; these were used in the same manner as the Stirling numbers of integer order are used to calcu-

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late the coefficients of a factorial series. Formulas for calculating $K_0(x)$ and $K_1(x)$ have been derived and programmed, using these modified Stirling numbers. Formulas for calculating $I_0(x)$ and $I_1(x)$ have been derived and programmed using series of incomplete beta functions in a similar algorithm. Results agree to thirteen significant figures for $K_0(x)$ and $K_1(x)$ when $x > 8$ and for $I_0(x)$ and $I_1(x)$ when $x > 15$. The modified Stirling numbers increase very slowly with order and index since gamma functions do not occur in the definition. Consequently no problems with overrun of the electronic computer occurred during the course of the calculations.

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I. INTRODUCTION

Factorial series for Bessel functions, confluent hypergeometric functions, and certain other special functions can be used to check the accuracy of calculations for these functions provided the argument is not too small. Generally, if a function is analytic in the right half plane, including the imaginary axis, and can be represented as a Laplace transform, then a factorial series can be derived which will converge in the right half plane. Buchal and Duffy[1] obtained factorial series for Hankel functions, Coulomb wave functions, and Mathieu functions. These authors also studied the convergence properties of the factorial series for the Hankel functions in considerable detail. Their results showed that factorial series were more accurate than Hankel asymptotic series for the same argument. The analysis was based on Bernoulli polynomials as discussed by Doetsch[2] and Milne-Thomson[3].

The analysis in this paper is based on an algorithm of Wasow[4] which uses Stirling numbers of the first kind to calculate coefficients for the factorial series. The Stirling numbers of the first kind increase very rapidly with order, eventually obtaining overrun in the electronic computer. Moreover, the factorial series for complex argument is awkward if there is a branch point at the origin. Rosser[5] obtained a generalized factorial series for modified

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[1] *R.N. Buchal and G. Duffy, "Factorial Series Representations as Check Solutions in the Development of Special Function Subroutines," Argonne National Laboratory Technical Memorandum No. 200, May 1970.*

[2] *G. Doetsch, <u>Hanbuch Der Laplace-Transformation, Band II,</u> Birkhauser Verlag, Bosel and Stuggart, 1955, pages 201-232.*

[3] *L.M. Milne - Thomson, <u>The Calculus of Finite Differences,</u> Macmillan and Co., Limited, London, 1933.*

[4] *W. Wasow, <u>Asymptotic Expansions for Ordinary Differential Equations,</u> Interscience Publishers, a Division of John Wiley & Sons, Inc., New York, pages 323-331.*

[5] *J. B. Rosser, "Factorial Expansions for Certain Bessel Functions," Mathematics Research Center Summary Report No. 1572, November 1975.*

Bessel functions of the second kind by direct manipulation of the Laplace transform and also established the convergence properties. His analysis is quite difficult and requires a separate treatment for each case. In this paper we derive Stirling numbers of the first kind and fractional order, leading to a generalization of Wasow's algorithm. Overrun in the computer is eliminated by scaling the Stirling numbers of integral and fractional order by omitting the gamma functions in the definition.

A further generalization of factorial series is required if the Laplace integral representing the function is evaluated between finite limits. If a factorial series is regarded as a series of beta functions, it is logical to represent a Laplace integral with finite limits in terms of a series of incomplete beta functions. The new series obtained by this method is convergent even though the corresponding factorial series may be divergent.

II. MODIFIED STIRLING NUMBERS OF THE FIRST KIND AND FRACTIONAL ORDER

Stirling numbers of the first kind are defined as coefficients which occur when a factorial is expanded into a polynomial[6,7], given by

$$x\,(x-1)\,(x-2)\,\ldots\,(x-n+1) = \sum_{m=o}^{n} S_n^{(m)}\, x^m . \tag{1}$$

Clearly m must be an integer in the above equation. However, a generating function involving logarithms is not subject to this restriction, such as

$$[\ell n\,(1+x)]^m = m! \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{n!}, \text{ for } |x|<1 \tag{2}$$

[6]M. Abramowitz and L.A. Stegun, Handbook of Mathematical Functions, Graphs, and Mathematical Tables. National Bureau of Standards, Washington, D.C., No. 55, Applied Mathematics Series, June 1964. See Formulas 24.1.3, page 824, for discussion of Stirling numbers. Continued fractions for the incomplete beta functions are given in Formulas 26.5.8, page 944.

[7]C. Jordan, Calculus of Finite Differences, Rottig and Romwalter, Sapron, Hungary, 1939. Stirling numbers are discussed in Chapter 4, pages 142-229.

or, alternatively,

$$[\ell n\ (1+x)]^m = \Gamma(m+1) \sum_{n=m}^{\infty} S_n^{(m)} \frac{x^n}{\Gamma(n+1)} .$$

In order to avoid the gamma function and to include non-integral indices we define $W_n^{(\upsilon)}$ by the equation

$$[\ell n\ (1+x)]^{\upsilon} = \sum_{k=o}^{\infty} W_k^{(\upsilon)}\ x^{\upsilon+k} , \tag{3}$$

where υ may take on fractional as well as integral values. When the $W_k^{(\nu)}$ have been calculated, Stirling numbers of the first kind may be obtained from the following equations.

$$S_n^{(m)} = \Gamma\ (n+1)\ W_{n-m}^{(m)}\ /\ \Gamma(m+1) , \tag{4}$$

where

$$\upsilon = m \tag{5}$$

$$k = n - m . \tag{6}$$

To find $W_o^{(\nu)}$, divide Eq. (3) by x^{ν} and evaluate the limit of each side of resulting equations as $x \to 0$. We obtain

$$W_o^{(\upsilon)} = 1. \tag{7}$$

We can also prove that

$$W_k^{(o)} = 0, \text{ for } k \geq 1 \tag{8}$$

and

$$W_o^{(o)} = 1. \tag{9}$$

We now derive a sequence of triangular equations for calculating $W_1^{(\upsilon)}$, $W_2^{(\upsilon)}$, ..., $W_\ell^{(\upsilon)}$ in turn. Let

$$y(x) = [\ell n\ (1+x)]^{\upsilon} \tag{10}$$

and

$$u(x) = (x+1)\ \ell n\ (1+x). \tag{11}$$

Then

$$u(x)\ y'(x) = \upsilon y(x)\ . \tag{12}$$

But

$$y(x) = \sum_{k=o}^{\infty} W_k^{(\upsilon)}\ x^{\upsilon+k}\ , \tag{13}$$

$$y'(x) = \sum_{k=o}^{\infty} W_k^{(\upsilon)}\ (\upsilon+k)\ x^{\upsilon+k-1} \text{ and} \tag{14}$$

$$u(x) = x + \tfrac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{1}{12} x^4 - \ldots + (-1)^k \frac{x^k}{k(k-1)} \text{, for } k>1. \tag{15}$$

On inserting these series into Eq. (12), carrying out the indicated multiplication and equating coefficients of like powers of x on each side of the resulting equation, we find

$$W_o^{(\upsilon)} = W_o^{(\upsilon)},$$

$$\frac{\upsilon+1}{1} W_1^{(\upsilon)} + \tfrac{1}{2} \upsilon = \upsilon W_1^{(\upsilon)} \text{ and}$$

$$\frac{\upsilon+2}{1} W_2^{(\upsilon)} + \frac{\upsilon+1}{2} W_1^{(\upsilon)} - \frac{\upsilon}{6} = \upsilon W_2^{(\upsilon)} .$$

On rearranging these equations we find

$$W_1^{(\upsilon)} + \tfrac{1}{2} \upsilon = 0 , \tag{16}$$

$$2 W_2^{(\upsilon)} + \frac{\upsilon+1}{2} W_1^{(\upsilon)} - \frac{\upsilon}{6} = 0 , \tag{17}$$

$$3 W_3^{(\upsilon)} + \frac{\upsilon+2}{2} W_2^{(\upsilon)} - \frac{\upsilon+1}{6} W_1^{(\upsilon)} + \frac{\upsilon}{12} = 0 , \tag{18}$$

and, in general,

$$\ell W_\ell^{(\upsilon)} + \sum_{k=0}^{\ell-1} (-1)^{\ell+k+1} \frac{v+k}{(\ell-k)(\ell-k+1)} W_k^{(\upsilon)} = 0. \tag{1\ )}$$

To find a recurrence formula involving different orders, differentiate Eq. (3) with respect to x and then multiply both sides of the resulting equation by (x+1), giving

$$\upsilon\ [\ln\ (1+x)]^{\upsilon-1} = \sum_{k=o}^{\infty} W_k^{(\upsilon)}\ (\upsilon+k)(x+1)\ x^{\upsilon+k-1}\ . \tag{20}$$

If we replace υ by $(\upsilon-1)$ in Eq. (3) and multiply both sides of the resulting equation by υ, we find

$$\upsilon\ [\ln(1+x)]^{\upsilon-1} = \sum_{k=o}^{\infty} W_k^{(\upsilon-1)}\ \upsilon x^{\upsilon+k-1}\ . \tag{21}$$

On comparing coefficients of like powers of x in the last two equations we find

$$W_{k+1}^{(\upsilon)} = [\upsilon\ W_{k+1}^{(\upsilon-1)} - (\upsilon+k)\ W_k^{(\upsilon)}]\ /\ [\upsilon+k+1]. \tag{22}$$

Hence we can calculate $W_{k+1}^{(\upsilon+1)}$ from $W_{k+1}^{(\upsilon)}$ and $W_k^{(\upsilon+1)}$. Higher order numbers can be generated in succession in the same manner. The values of $W_o^{(\upsilon)}$, $W_1^{(\upsilon)}$ ..., $W_\ell^{(\upsilon)}$ must be calculated from Eq. (19) before the recurrence formula given by Eq. (22) can be used; a double entry table is finally obtained.

Wasow uses Schlomlich's definition of factorial coefficients in his development of factorial series [4]

$$x\ (x+1)\ (x+2)\ \ldots\ (x+n-1) = \sum_{m=o}^{n-1} \Gamma_m^{\ n}\ x^{n-m}\ . \tag{23}$$

It follows that the factorial coefficients and Stirling numbers of the first kind are related by the formula

$$\Gamma^{n}_{n-m} = (-1)^{n-m} S^{(m)}_{n} . \tag{24}$$

We define

$$V^{(m)}_{n-m} = (-1)^{m-n} \Gamma(m+1) S^{(m)}_{n} / \Gamma(n+1) \tag{25}$$

and for fractional orders

$$[- \ell n(1-x)]^{\upsilon} = \sum_{k=o}^{\infty} V^{(\upsilon)}_{k} x^{\upsilon+k}. \tag{26}$$

Finally, we obtain the following recurrence formulas in the manner indicated previously

$$\ell V^{(\upsilon)}_{\ell} - \sum_{k=o}^{\ell-1} \frac{\upsilon+k}{(\ell-k)(\ell-k+1)} V^{(\upsilon)}_{k} = 0 \text{ and} \tag{27}$$

$$V^{(\upsilon)}_{k+1} = [\upsilon V^{(\upsilon-1)}_{k+1} + (\upsilon+k) V^{(\upsilon)}_{k}] / [\upsilon+k+1] . \tag{28}$$

III. GENERALIZED FACTORIAL SERIES

We now derive an extension of Wasow's algorithm for a Laplace integral to functions with a branch point at the origin. The branch point involves fractional powers, in the same context as Watson's Lemma[8]; logarithmic branch points are not considered. Assume

---

[8] *E.T. Copson, <u>An Introduction to the Theory of Functions of a Complex Variable</u>, Oxford, 1935, pages 218-219.*

$$F(x) = \int_0^\infty f(t)\, e^{-xt}\, dt \tag{29}$$

and let

$$t = -\ln(1-u)\ ; \tag{30}$$

then

$$F(x) = \int_0^1 f[-\ln(1-u)]\ [1-u]^{x-1} du\ . \tag{31}$$

If

$$f(t) = \sum_{n=o}^{\infty} a_n t^{\upsilon+n}, \text{ for } \upsilon>-1 \text{ and } 0<t<1\ , \tag{32}$$

then

$$F(x) = \sum_{n=o}^{\infty} a_n \int_0^1 [-\ln(1-u)]^{\upsilon+n} [1-u]^{x-1} du. \tag{33}$$

On referring to Eq. (26) we see

$$F(x) = \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n V_k^{(\upsilon+n)} \int_0^1 u^{\upsilon+k+n} (1-u)^{x-1} du. \tag{34}$$

Now

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1} (1-t)^{\beta-1} dt \tag{35}$$

so that

$$F(x) = \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n V_k^{(\upsilon+n)} B(k+n+\upsilon+1,x) \; . \tag{36}$$

We enter the terms in a double entry table; then by summing the diagonals, we obtain the Cauchy sum of the double series[9]. Let

$$k+n=\ell \tag{37}$$

and define

$$b_\ell = \sum_{n=o}^{\ell} a_n V_{\ell-n}^{(\upsilon+n)} \; ; \tag{38}$$

then

$$F(x) = \sum_{\ell=o}^{\infty} b_\ell B(\upsilon+\ell+1,x) \; . \tag{39}$$

On noting that

$$B(\upsilon+\ell+1,x) = B(\upsilon,x) \frac{\upsilon(\upsilon+1)\ldots(\upsilon+\ell)}{(x+\upsilon)(x+\upsilon+1)\ldots(x+\upsilon+\ell)} \tag{40}$$

---

[9] *K. Knapp, Infinite Sequences and Series, Dover Publications, New York, 1956, pages 82-85.*

and using Pochhammer's symbol to represent the factorials, we find

$$F(x) = B(\upsilon,x) \sum_{\ell=0}^{\infty} b_\ell (\upsilon)_{\ell+1} / (x+\upsilon)_{\ell+1} \tag{41}$$

in a formal sense. By analogy with conventional factorial series, the series should converge in a half plane which lies to the right of the imaginary axis. Details of the required analysis will not be considered at this time.

IV. A SERIES OF INCOMPLETE BETA FUNCTIONS

Since a generalized factorial series is in fact a series of beta functions, it is natural to represent a Laplace integral with finite limits of integration as a series of incomplete beta functions. The lower limit of integration can be taken equal to zero without loss of generality. Assume

$$F(x) = \int_0^\tau e^{-tx} f(t)\, dt, \; t>0 . \tag{42}$$

Let

$$\varepsilon = 1-e^{-\tau} ; \tag{43}$$

then

$$F(x) = \int_0^\varepsilon f[-\ell n(1-u)]\, [1-u]^{x-1} du . \tag{44}$$

On referring to Eq. (26) we find

$$F(x) = \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n V_k^{(\nu+n)} \int_o^{\varepsilon} u^{\nu+k+n} (1-u)^{x-1} du \;. \tag{45}$$

Since

$$B_{\varepsilon}(\alpha,\beta) = \int_o^{\varepsilon} t^{\alpha-1} (1-t)^{\beta-1} dt \;, \tag{46}$$

we find

$$F(x) = \sum_{n=o}^{\infty} \sum_{k=o}^{\infty} a_n V_k^{(\nu+n)} B_{\varepsilon}(\nu+k+n+1,x) \tag{47}$$

or

$$F(x) = \sum_{\ell=o}^{\infty} b_{\ell} B_{\varepsilon}(\nu+\ell+1,x) \tag{48}$$

on referring to Eqs. (37) and (38).

To compute the incomplete beta function, set

$$\nu+\ell = \delta \tag{49}$$

and use the formula

$$B_{\varepsilon}(\delta+1,x) = B(x,\delta+1) - B_{1-\varepsilon}(x,\delta+1) \;. \tag{50}$$

The beta function on the right side of Eq. (50) was expressed in terms of gamma functions, as shown in Eq. (51),

$$B(x,\delta+1) = \Gamma(x)\ \Gamma(\delta+1)\ /\ \Gamma(x+\delta+1). \tag{51}$$

The gamma functions were obtained from the subroutine CDLGAM by H. Kuki[10]. This subroutine is valid for both real and complex values of x.

The incomplete beta function on the right side of Eq. (50) is given by the integral formula

$$B_{1-\varepsilon}(x,\ \delta+1) = \int_0^{1-\varepsilon} t^{x-1}\ (1-t)^{\delta} dt. \tag{52}$$

On expanding the binomial factor in the integrand and integrating term by term we find

$$B_{1-\varepsilon}(x,\delta+1) = \frac{(1-\varepsilon)^x}{x} - \frac{\delta}{1!}\ \frac{(1-\varepsilon)^{x+1}}{(x+1)}$$

$$+ \frac{\delta(\delta-1)}{2!}\ \frac{(1-\varepsilon)^{x+2}}{(x+2)} - \frac{\delta(\delta-1)(\delta-2)}{3!}\ \frac{(1-\varepsilon)^{x+3}}{(x+3)} + \ldots \tag{53}$$

This series is satisfactory if δ is small, but is subject to round-off error if δ is large and positive. To overcome this difficulty, integrate the right-hand side of Eq. (52) repeatedly by parts. We find

$$B_{1-\varepsilon}(x,\delta+1) = \frac{(1-\varepsilon)^x}{x}\ \varepsilon^{\delta} + \frac{\delta(1-\varepsilon)^{x+1}}{x(x+1)}\ \varepsilon^{\delta-1}$$

$$+ \frac{\delta(\delta-1)(1-\varepsilon)^{x+2}}{x(x+1)(x+2)}\ \varepsilon^{\delta-2} + \ldots + \frac{\delta(\delta-1)(\delta-2)\ldots(\delta-m)}{x(x+1)(x+2)\ldots(x+m)}\ R_m\ , \tag{54}$$

where

---

[10] *H. Kuki, Communications of the ACM, Vol 15, No 4, April 1972. The subroutine CDLGAM was extracted from Algorithm 421.*

$$R_m = B_{1-\varepsilon}\,(x+m+1,\ \delta-m)\ . \tag{55}$$

We choose m so that

$$1 < \delta-m \leq 2. \tag{56}$$

Then the terms of the series given by Eq. (54) are positive when x is real and positive, and consequently the round-off error should be small.

The remainder R_m is calculated from Eq. (53),

$$R_m = \frac{(1-\varepsilon)^{x+m+1}}{x+m+1} - \frac{(\delta-m-1)(1-\varepsilon)^{x+m+2}}{1!\ (x+m+2)} + \frac{(\delta-m-1)(\delta-m-2)(1-\varepsilon)^{x+m+3}}{2!\ (x+m+3)} - \cdots \tag{57}$$

All the terms of this series after the first term are negative, and decrease rapidly in magnitude. Hence, the error in calculating R_m should be small.

V. CALCULATIONS

We calculated the modified Bessel functions $K_o(x)$, $K_1(x)$, $I_o(x)$, and $I_1(x)$ from the integrals

$$K_o(x) = \frac{\Gamma(1/2)\ e^{-x}}{\Gamma(1/2)} \int_o^{\infty} e^{-xt}(t^2+2t)^{-1/2}\,dt\ , \tag{58}$$

$$K_1(x) = \frac{\Gamma(1/2)\ (1/2\,x)e^{-x}}{\Gamma(\frac{3}{2})} \int_o^{\infty} e^{-xt}(t^2+2t)^{1/2}\,dt, \tag{59}$$

$$I_o(x) = \frac{e^x}{\Gamma(1/2)\Gamma(1/2)} \int_o^{2} e^{-xt}(2t-t^2)^{-1/2}\,dt, \tag{60}$$

$$I_1(x) = \frac{(1/2\,x)\ e^x}{\Gamma(1/2)\ \Gamma(\frac{3}{2})} \int_o^{2} e^{-xt}(2t-t^2)^{1/2}\,dt. \tag{61}$$

On referring to Eqs. (29) and (32), we see that

$$\nu = -\tfrac{1}{2} \qquad (62)$$

in Eqs. (58) and (60), and

$$\nu = \tfrac{1}{2} \qquad (63)$$

in Eqs. (59) and (61). Hence, the modified Stirling numbers $V_k^{-1/2+m}$ and $V_k^{1/2+m}$ are required for the coefficients of the factorial series. A short table of these numbers is given in Table 1.

Next, the coefficients a_n for the series expansions of $(1+\tfrac{1}{2}t)^{-1/2}$, $(1-\tfrac{1}{2}t)^{-1/2}$, $(1+\tfrac{1}{2}t)^{1/2}$ and $(1-\tfrac{1}{2}t)^{1/2}$ were calculated from the appropriate recurrence formulas, as shown in Table 2.

The coefficients b_ℓ were calculated from Eq. (38) for each of the four cases listed above. In addition, the partial sums

$$C_m = \sum_{\ell=0}^{m} b_\ell \qquad (64)$$

were calculated in order to study the convergence of the factorial series. The series $\sum_m^{\infty} C_m$ must converge if the corresponding factorial series is to converge. This condition is apparently violated for the functions $I_0(x)$ and $I_1(x)$, which shows why the factorial series for these functions apparently diverged. These results are given in Table 3.

The beta functions and incomplete beta functions required in the series expansions for $K_0(x)$, $K_1(x)$, $I_0(x)$, and $I_1(x)$ were calculated from formulas discussed previously. Sample tabulations are shown in Table 4. Finally, the modified Bessel functions were calculated for a limited range of variables. These results are shown in Table 5.

TABLE 1. MODIFIED STIRLING NUMBERS, $V_{\kappa}^{(\nu)}$

ν \ κ	0	1	2	3	4	5	
-.5	1.000	-.250	-.073	-.039	-.026	-.019	...
.5	1.000	.250	.135	.091	.058	.054	...
1.5	1.000	.750	.594	.492	.421	.369	...
2.5	1.000	1.250	1.302	1.289	1.253	1.209	...
3.5	1.000	1.750	2.260	2.607	2.844	3.007	...
4.5	1.000	2.250	3.469	4.570	5.538	6.380	...
5.5	1.000	2.750	4.927	7.305	9.742	12.156	...
6.5	1.000	3.250	6.635	10.935	15.924	21.410	...
7.5	1.000	3.750	8.594	15.386	24.616	35.493	...
8.5	1.000	4.250	10.802	21.383	36.411	56.061	...
9.5	1.000	4.750	13.260	28.451	51.966	85.112	...
10.5	1.000	5.250	15.969	36.914	71.999	125.016	...
11.5	1.000	5.750	18.927	46.898	97.291	178.523	...
12.5	1.000	6.250	22.135	58.529	128.687	243.849	...
13.5	1.000	6.750	25.594	71.930	167.092	339.653	...
14.5	1.000	7.250	29.302	87.227	213.476	455.091	...
15.5	1.000	7.750	33.260	104.544	268.869	599.645	...
16.5	1.000	8.250	37.469	124.008	334.365	779.160	...
17.5	1.000	8.750	41.927	145.742	411.121	998.863	...
18.5	1.000	9.250	46.635	169.672	500.356	1265.403	...
19.5	1.000	9.750	51.594	196.523	603.349	1585.680	...
20.5	1.000	10.250	56.802	225.920	721.447	1968.079	...
21.5	1.000	10.750	62.260	257.868	856.053	2420.492	...
22.5	1.000	11.250	67.969	292.852	1008.638	2952.363	...
23.5	1.000	11.750	73.927	330.836	1180.733	3573.708	...

TABLE 2. BINOMIAL COEFFICIENTS FOR SERIES EXPANSION

N	$(1+.5*T)^{-.5}$	$(1+.5*T)^{.5}$	$(1-.5*T)^{-.5}$	$(1-.5*T)^{.5}$
1	.1000000000D+01	.1000000000D+01	.1000000000D+01	.1000000000D+01
2	.2500000000D+00	-.2500000000D+00	-.2500000000D+00	.2500000000D+00
3	.9375000000D-01	-.3125000000D-01	.9375000000D-01	-.3125000000D-01
4	.3906250000D-01	-.7812500000D-02	-.3906250000D-01	.7812500000D-02
5	.1708984375D-01	-.2441406250D-02	.1708984375D-01	-.2441406250D-02
6	.7690429688D-02	-.8544921875D-03	-.7690429688D-02	.8544921875D-03
7	.3524780273D-02	-.3204345703D-03	.3524780273D-02	-.3204345703D-03
8	.1636505127D-02	-.1258850098D-03	-.1636505127D-02	.1258850098D-03
9	.7671117783D-03	-.5114078522D-04	.7671117783D-03	-.5114078522D-04
10	.3622472286D-03	-.2130866051D-04	-.3622472286D-03	.2130866051D-04
11	.1720674336D-03	-.9056180716D-05	.1720674336D-03	-.9056180716D-05
12	.8212309331D-04	-.3910623491D-05	-.8212309331D-04	.3910623491D-05
13	.3935064888D-04	-.1710897777D-05	.3935064888D-04	-.1710897777D-05
14	.1891858119D-04	-.7567432476D-06	-.1891858119D-04	.7567432476D-06
15	.9121458788D-05	-.3378318070D-06	.9121458788D-05	-.3378318070D-06
16	.4408705081D-05	-.1520243131D-06	-.4408705081D-05	.1520243131D-06
17	.2135466524D-05	-.6888601689D-07	.2135466524D-05	-.6888601689D-07
18	.1036329342D-05	-.3140391946D-07	-.1036329342D-05	.3140391946D-07
19	.5037712081D-06	-.1439346309D-07	.5037712081D-06	-.1439346309D-07
20	.2452570355D-06	-.6628568527D-08	-.2452570355D-06	.6628568527D-08
21	.1195628048D-06	-.3065712944D-08	.1195628048D-06	-.3065712944D-08
22	.5835803568D-07	-.1423366724D-08	-.5835803568D-07	.1423366724D-08
23	.2851585834D-07	-.6631594964D-09	.2851585834D-07	-.6631594964D-09
24	.1394797419D-07	-.3099549820D-09	-.1394797419D-07	.3099549820D-09
25	.6828695697D-08	-.1452913978D-09	.6828695697D-08	-.1452913978D-09

TABLE 3. PARTIAL SUMS, C_M

M	C_M (FOR $I_0(X)$)	C_M (FOR $I_1(X)$)	C_M (FOR $K_0(X)$)	C_M (FOR $K_1(X)$)
1	.1000000000D+01	.1000000000D+01	.1000000000D+01	.1000000000D+01
2	0.	0.	-.5000000000D+00	.5000000000D+00
3	.8333333333D-01	-.8333333333D-01	-.4166666667D-01	.2916666667D+00
4	.1041666667D+00	-.1041666667D+00	-.4166666667D-01	.2033333333D+00
5	.1187500000D+00	-.1118055556D+00	-.2447916667D-01	.1616319444D+00
6	.1329861111D+00	-.1156250000D+00	-.1814236111D-01	.1320312500D+00
7	.1483775628D+00	-.1183139054D+00	-.1392712457D-01	.1115689071D+00
8	.1655691964D+00	-.1206835401D+00	-.1121135086D-01	.9653663155D-01
9	.1849897015D+00	-.1235965120D+00	-.9297019245D-02	.8514439389D-01
10	.2070163777D+00	-.1267699119D+00	-.7891392514D-02	.7612111033D-01
11	.2220438054D+00	-.1304793276D+00	-.6820430172D-02	.6832367917D-01
12	.2605000552D+00	-.1347898414D+00	-.5981169633D-02	.6230056847D-01
13	.2928672853D+00	-.1397599840D+00	-.5308128754D-02	.5774504858D-01
14	.3296929445D+00	-.1454462447D+00	-.4757997793D-02	.5344147367D-01
15	.3716013404D+00	-.1519085092D+00	-.4301053015D-02	.4973381885D-01
	.			
	.			
	.			
85	.4486264325D+04	-.2132930456D+03	-.4049428749D-03	.8466639314D-02
86	.5158981027D+04	-.2421390452D+03	-.3988377917D-03	.8367300496D-02
87	.5932950354D+04	-.2749495588D+03	-.3928963534D-03	.8270263456D-02
88	.6823459192D+04	-.3122761345D+03	-.3871123161D-03	.8175449146D-02
89	.7848108406D+04	-.3547478056D+03	-.3814797434D-03	.8032782095D-02
90	.9027163886D+04	-.4030820658D+03	-.3759929874D-03	.7992190200D-02

TABLE 4. INCOMPLETE AND COMPLETE BETA FUNCTIONS

X = 15.   ε = .36456

ν+ℓ+1	INCOMPLETE	COMPLETE
.50	.4614745534D+00	.4614745534D+00
1.50	.1488627592D-01	.1488627592D-01
2.50	.1353297811D-02	.1353297811D-02
3.50	.1933282586D-03	.1933282587D-03
4.50	.3657561550D-04	.3657561650D-04
5.50	.8440525882D-05	.8440526885D-05
6.50	.2264531600D-05	.2264531603D-05
7.50	.6846258310D-06	.6846258336D-06
8.50	.2282086089D-06	.2282086112D-06
9.50	.6254353824D-07	.6254354022D-07
10.50	.3200667713D-07	.3200667886D-07
11.50	.1317921920D-07	.1317922071D-07
12.50	.5719283140D-08	.5719284458D-08
13.50	.2599673602D-08	.2599674753D-08
14.50	.1231423877D-08	.1231424883D-08
15.50	.6052757576D-09	.6052766375D-09
16.50	.3075988334D-09	.3075996027D-09
17.50	.1611229288D-09	.1611236014D-09
18.50	.8675827411D-10	.8675886229D-10
19.50	.4791109612D-10	.4791151052D-10
20.50	.2708002559D-10	.2708047551D-10
21.50	.1563762750D-10	.1563802107D-10
22.50	.9211092752D-11	.9211437068D-11
23.50	.5525560982D-11	.5526862241D-11
24.50	.3373275674D-11	.3373539290D-11

TABLE 5. MODIFIED BESSEL FUNCTIONS

X	$I_0(X)$	$I_1(X)$	$K_0(X)$	$K_1(X)$	
15.	.3396493733D+05	.3281249220D+06	.9319536482D-07	.1014172937D-06	IBF, FOSN
15.	.3396493733D+05	.3281249220D+06	.9319536482D-07	.1014172937D-06	SUBROUTINE
20.	.4355828255D+08	.4245497339D+08	.5741237815D-09	.5883057970D-09	IBF, FOSN
20.	.4355828255D+08	.4245497339D+08	.5741237815D-09	.5883057970D-09	SUBROUTINE
25.	.5774560606D+10	.5657865130D+10	.3464161562D-11	.3532778073D-11	IBF, FOSN
25.	.5774560606D+10	.5657865130D+10	.3464161562D-11	.3532778073D-11	SUBROUTINE
30.	.7816722978D+12	.7685320389D+12	.2132477496D-13	.2167732002D-13	IBF, FOSN
30.	.7816722978D+12	.7685320389D+12	.2132477496D-13	.2167732002D-13	SUBROUTINE
35.	.1073388185D+15	.1057941261D+15	.1331035149D-15	.1349917334D-15	IBF, FOSN
35.	.1073388185D+15	.1057941261D+15	.1331035149D-15	.1349917334D-15	SUBROUTINE
40.	.1489477479D+17	.1470739616D+17	.8392861100D-18	.8497131955D-18	IBF, FOSN
40.	.1489477479D+17	.1470739616D+17	.8392861100D-18	.8497131955D-18	SUBROUTINE
45.	.2083414075D+19	.2060133462D+19	.5333456123D-20	.5392394594D-20	IBF, FOSN
45.	.2083414075D+19	.2060133462D+19	.5333456123D-20	.5392394594D-20	SUBROUTINE
50.	.2932553784D+21	.2903078590D+21	.3410167750D-22	.3444102227D-22	IBF, FOSN
50.	.2932553784D+21	.2903078590D+21	.3410167750D-22	.3444102227D-22	SUBROUTINE
55.	.4148789561D+23	.4110898645D+23	.2191310218D-24	.2211142272D-24	IBF, FOSN
55.	.4148789561D+23	.4110898645D+23	.2191310218D-24	.2211142272D-24	SUBROUTINE
60.	.5894077056D+25	.5844751588D+25	.1413897841D-26	.1425632027D-26	IBF, FOSN
60.	.5894077056D+25	.5844751588D+25	.1413897841D-26	.1425632027D-26	SUBROUTINE
65.	.8403039846D+27	.8338148547D+27	.9154467321D-29	.9224619528D-29	IBF, FOSN
65.	.8403039846D+27	.8338148547D+27	.9154467321D-29	.9224619528D-29	SUBROUTINE
70.	.1201588955D+30	.1192975079D+30	.5944661337D-31	.5986973674D-31	IBF, FOSN
70.	.1201588955D+30	.1192975079D+30	.5944661337D-31	.5986973674D-31	SUBROUTINE
75.	.1722639079D+32	.1711116015D+32	.3870117046D-33	.3895832947D-33	IBF, FOSN
75.	.1722639079D+32	.1711116015D+32	.3870117046D-33	.3895832947D-33	SUBROUTINE

VI. RESULTS AND CONCLUSIONS

These series expansions in terms of beta functions and incomplete beta functions were derived in order to check the accuracy of our Bessel function subroutine[11,12] with independent calculations. The error analysis of our subroutine by theoretical methods would be very difficult, especially for the section involving continued fractions. Hence, computational efficiency is a secondary consideration for the new series expansions. Addressing the comment of the reviewer*, we believe our algorithm for the incomplete beta function is as efficient as the continued fractions of Stegun[6] when x is large, since Eq. (54) can then be used without the remainder. We have not made any specific comparison for small values of x.

Since factorial series are a method of summing certain asymptotic series, they are most effective for large and moderately large values of the argument. The convergence is slow when x is small, so that an excessive number of terms is required. Round-off error may occur in the coefficients b_ℓ and in summing the series. Alternate methods of calculating Bessel functions, such as quadratures, are required when x is small. Subroutines used for checking should not use continued fractions or other procedures used in the subroutine, to insure the calculations are in fact independent.

The generalized factorial series for $K_0(x)$ and $K_1(x)$ yield accurate numerical values when x is only moderately large and the Hankel asymptotic series is not sufficiently accurate. On the other hand, the new series for $I_0(x)$ and $I_1(x)$ have the same range of accuracy as the Hankel asymptotic series, and do not offer any computational advantage. This is probably due to the apparent divergence of the series for the partial sums of b_ℓ.

The generalized factorial series for $K_0(x)$ and $K_1(x)$ are being extended to the complex plane. Programming of generalized factorial series for the ordinary Bessel functions is in progress. Alternate methods of calculating

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[11] A. S. Elder, *"Formulas for Calculating Bessel Functions of Integral Order and Complex Argument," BRL Report No. 1423, November, 1968.*

[12] K.L. Zimmerman, A.S. Elder and A.K. Depue, *"User's Manual for the BRL Subroutine to Calculate Bessel Functions of Integral Order and Complex Argument," ARBRL-TR-02068, May 1978.*

*See ACKNOWLEDGMENTS

$I_n(x)$ are also being considered, as the results obtained in this paper fell short of our expectations.

The modified Stirling numbers, as defined in this paper, are more useful for computations involving Wasow's algorithm than the original Stirling numbers, as problems arising from very large numbers and overrun of the computer registers are entirely avoided. The method of scaling employed in this paper, which merely involves the omission of gamma functions in the definition of Stirling numbers, is more effective than the method of scaling used in previous paper by the authors[13].

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[13] *A.S. Elder and E.M. Wineholt, "Factorial and Hadamard Series for Bessel Functions of Orders Zero and One," Transactions of the Twenty-Second Conference of Army Mathematicians, Army Research Office, 1977, pp. 277- 287.*

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10. H. Kuki, Communications of the ACM, Vol 15, No. 4, April 1972. The subroutine CDLGAM was extracted from Algorithm 421.

11. A. S. Elder, "Formulas for Calculating Bessel Functions of Integral Order and Complex Argument," BRL Report No. 1423, November, 1968.

12. K.L. Zimmerman, A.S. Elder and A.K. Depue, "User's Manual for the BRL Subroutine to Calculate Bessel Functions of Integral Order and Complex Argument," ARBRL-TR-02068, May 1978.

13. A.S. Elder and E.M. Wineholt, "Factorial and Hadamard Series for Bessel Functions of Orders Zero and One," Transactions of the Twenty-Second Conference of Army Mathematicians, Army Research Office, 1977, pp 277-287.

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